Power-Series Solutions

In this section we extend our study of second-order linear homogeneous equations with variable coefficients. With the Euler equations in Section 16.4, the power of the variable \( x \) in the nonconstant coefficient had to match the order of the derivative with which it was paired: \( x^2 \) with \( y'' \), \( x^3 \) with \( y' \), and \( x^0 (=1) \) with \( y \). Here we drop that requirement so we can solve more general equations.

**Method of Solution**

The power-series method for solving a second-order homogeneous differential equation consists of finding the coefficients of a power series

\[
y(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots
\]

which solves the equation. To apply the method we substitute the series and its derivatives into the differential equation to determine the coefficients. The technique for finding the coefficients is similar to that used in the method of undetermined coefficients presented in Section 16.2.

In our first example we demonstrate the method in the setting of a simple equation whose general solution we already know. This is to help you become more comfortable with solutions expressed in series form.

**EXAMPLE 1** Solve the equation \( y'' + y = 0 \) by the power-series method.

**Solution** We assume the series solution takes the form of

\[
y = \sum_{n=0}^{\infty} c_n x^n
\]

and calculate the derivatives

\[
y' = \sum_{n=1}^{\infty} n c_n x^{n-1} \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}.
\]

Substitution of these forms into the second-order equation gives us

\[
\sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} + \sum_{n=0}^{\infty} c_n x^n = 0.
\]

Next, we equate the coefficients of each power of \( x \) to zero as summarized in the following table.

<table>
<thead>
<tr>
<th>Power of ( x )</th>
<th>Coefficient Equation</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x^0 )</td>
<td>( 2(1)c_2 + c_0 = 0 ) or ( c_2 = \frac{1}{2} c_0 )</td>
</tr>
<tr>
<td>( x^1 )</td>
<td>( 3(2)c_1 + c_1 = 0 ) or ( c_1 = \frac{1}{3} c_1 )</td>
</tr>
<tr>
<td>( x^2 )</td>
<td>( 4(3)c_4 + c_2 = 0 ) or ( c_4 = \frac{1}{4} c_2 )</td>
</tr>
<tr>
<td>( x^3 )</td>
<td>( 5(4)c_5 + c_3 = 0 ) or ( c_5 = \frac{1}{5} c_3 )</td>
</tr>
<tr>
<td>( x^4 )</td>
<td>( 6(5)c_6 + c_4 = 0 ) or ( c_6 = \frac{1}{6} c_4 )</td>
</tr>
<tr>
<td>( \vdots )</td>
<td>( \vdots )</td>
</tr>
<tr>
<td>( x^{n-2} )</td>
<td>( n(n-1)c_n + c_{n-2} = 0 ) or ( c_n = -\frac{1}{n(n-1)} c_{n-2} )</td>
</tr>
</tbody>
</table>
From the table we notice that the coefficients with even indices \((n = 2k, k = 1, 2, 3, \ldots)\) are related to each other and the coefficients with odd indices \((n = 2k + 1)\) are also inter-related. We treat each group in turn.

**Even indices:** Here \(n = 2k\), so the power is \(x^{2k-2}\). From the last line of the table, we have
\[
2k(2k - 1)c_{2k} + c_{2k-2} = 0
\]
or
\[
c_{2k} = -\frac{1}{2k(2k - 1)}c_{2k-2}.
\]

From this recursive relation we find
\[
c_{2k} = \left[ -\frac{1}{2k(2k - 1)} \right] \left[ -\frac{1}{(2k - 2)(2k - 3)} \right] \cdots \left[ -\frac{1}{4(3)} \right] \left[ -\frac{1}{2} \right] c_0
\]
\[
= \frac{(-1)^k}{(2k)!} c_0.
\]

**Odd indices:** Here \(n = 2k + 1\), so the power is \(x^{2k-1}\). Substituting this into the last line of the table yields
\[
(2k + 1)(2k)c_{2k+1} + c_{2k-1} = 0
\]
or
\[
c_{2k+1} = -\frac{1}{(2k + 1)(2k)}c_{2k-1}.
\]

Thus,
\[
c_{2k+1} = \left[ \frac{1}{(2k + 1)(2k)} \right] \left[ \frac{1}{(2k - 1)(2k - 2)} \right] \cdots \left[ \frac{1}{5(4)} \right] \left[ \frac{1}{3(2)} \right] c_1
\]
\[
= \frac{(-1)^k}{(2k + 1)!} c_1.
\]

Writing the power series by grouping its even and odd powers together and substituting for the coefficients yields
\[
y = \sum_{n=0}^{\infty} c_n x^n
\]
\[
= \sum_{k=0}^{\infty} c_{2k} x^{2k} + \sum_{k=0}^{\infty} c_{2k+1} x^{2k+1}
\]
\[
= c_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} + c_1 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k + 1)!} x^{2k+1}.
\]

From Table 8.1 in Section 8.10, we see that the first series on the right-hand side of the last equation represents the cosine function and the second series represents the sine. Thus, the general solution to \(y'' + y = 0\) is
\[
y = c_0 \cos x + c_1 \sin x.
\]
EXAMPLE 2  Find the general solution to $y'' + xy' + y = 0$.

Solution  We assume the series solution form

$$y = \sum_{n=0}^{\infty} c_n x^n$$

and calculate the derivatives

$$y' = \sum_{n=1}^{\infty} n c_n x^{n-1} \quad \text{and} \quad y'' = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2}.$$

Substitution of these forms into the second-order equation yields

$$\sum_{n=2}^{\infty} n(n-1)c_n x^{n-2} + \sum_{n=1}^{\infty} n c_n x^n + \sum_{n=0}^{\infty} c_n x^n = 0.$$

We equate the coefficients of each power of $x$ to zero as summarized in the following table.

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<thead>
<tr>
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<td>$2(1)c_2 \quad + c_0 = 0 \quad \text{or} \quad c_2 = -\frac{1}{2}c_0$</td>
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<tr>
<td>$x^1$</td>
<td>$3(2)c_3 + c_1 + c_1 = 0 \quad \text{or} \quad c_3 = -\frac{1}{3}c_1$</td>
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<tr>
<td>$x^2$</td>
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<tr>
<td>$x^3$</td>
<td>$5(4)c_5 + 3c_3 + c_3 = 0 \quad \text{or} \quad c_5 = -\frac{1}{5}c_3$</td>
</tr>
<tr>
<td>$x^4$</td>
<td>$6(5)c_6 + 4c_4 + c_4 = 0 \quad \text{or} \quad c_6 = -\frac{1}{6}c_4$</td>
</tr>
<tr>
<td>$\vdots$</td>
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</tr>
<tr>
<td>$x^n$</td>
<td>$(n+2)(n+1)c_{n+2} + (n+1)c_n = 0 \quad \text{or} \quad c_{n+2} = -\frac{1}{n+2}c_n$</td>
</tr>
</tbody>
</table>

From the table notice that the coefficients with even indices are interrelated and the coefficients with odd indices are also interrelated.

**Even indices:** Here $n = 2k - 2$, so the power is $x^{2k-2}$. From the last line in the table, we have

$$c_{2k} = -\frac{1}{2k}c_{2k-2}.$$

From this recurrence relation we obtain

$$c_{2k} = \left(-\frac{1}{2k}\right)\left(-\frac{1}{2k-2}\right)\cdots\left(-\frac{1}{6}\right)\left(-\frac{1}{4}\right)\left(-\frac{1}{2}\right)c_0$$

$$= \frac{(-1)^k}{(2)(4)(6)\cdots(2k)}c_0.$$

**Odd indices:** Here $n = 2k - 1$, so the power is $x^{2k-1}$. From the last line in the table, we have

$$c_{2k+1} = -\frac{1}{2k+1}c_{2k-1}.$$

From this recurrence relation we obtain

$$c_{2k+1} = \left(-\frac{1}{2k+1}\right)\left(-\frac{1}{2k-1}\right)\cdots\left(-\frac{1}{3}\right)\left(-\frac{1}{1}\right)c_1$$

$$= \frac{(-1)^k}{(3)(5)\cdots(2k+1)}c_1.$$
Writing the power series by grouping its even and odd powers and substituting for the coefficients yields

\[ y = \sum_{k=0}^{\infty} c_{2k}x^{2k} + \sum_{k=0}^{\infty} c_{2k+1}x^{2k+1} = c_0 \sum_{k=0}^{\infty} \frac{(-1)^k}{(2)(4)\cdots(2k)} x^{2k} + c_1 \sum_{k=0}^{\infty} \frac{(-1)^k}{(3)(5)\cdots(2k+1)} x^{2k+1}. \]

**EXAMPLE 3** Find the general solution to

\[(1 - x^2)y'' - 6xy' - 4y = 0, \quad |x| < 1.\]

**Solution** Notice that the leading coefficient is zero when \(x = \pm 1\). Thus, we assume the solution interval \(I: -1 < x < 1\). Substitution of the series form

\[ y = \sum_{n=0}^{\infty} c_n x^n \]

and its derivatives gives us

\[(1 - x^2) \sum_{n=2}^{\infty} n(n - 1)c_n x^{n-2} - 6 \sum_{n=1}^{\infty} n c_n x^n - 4 \sum_{n=0}^{\infty} c_n x^n = 0,\]

\[ \sum_{n=2}^{\infty} n(n - 1)c_n x^{n-2} - 6 \sum_{n=1}^{\infty} n c_n x^n - 4 \sum_{n=0}^{\infty} c_n x^n = 0. \]

Next, we equate the coefficients of each power of \(x\) to zero as summarized in the following table.

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<td>(\vdots)</td>
</tr>
<tr>
<td>(x^n)</td>
<td>((n + 2)(n + 1)c_{n+2} - [n(n - 1) + 6n + 4]c_n = 0)</td>
</tr>
<tr>
<td>(\vdots)</td>
<td>(\vdots)</td>
</tr>
<tr>
<td>(n(n + 1)c_{n+2} - (n + 4)(n + 1)c_n = 0) or (c_{n+2} = \frac{n + 4}{n + 2}c_n)</td>
<td></td>
</tr>
</tbody>
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Again we notice that the coefficients with even indices are interrelated and those with odd indices are interrelated.

*Even indices:* Here \(n = 2k - 2\), so the power is \(x^{2k}\). From the right-hand column and last line of the table, we get

\[ c_{2k} = \frac{2k + 2}{2k} c_{2k-2} = \left( \frac{2k + 2}{2k} \right) \left( \frac{2k}{2k - 2} \right) \left( \frac{2k - 2}{2k - 4} \right) \cdots \left( \frac{4}{2} \right) \cdot 0 = (k + 1)c_0. \]
Odd indices: Here $n = 2k - 1$, so the power is $x^{2k+1}$. The right-hand column and last line of the table gives us

\[
c_{2k+1} = \frac{2k + 3}{2k + 1} c_{2k-1}
\]

\[
= \left( \frac{2k + 3}{2k + 1} \right) \left( \frac{2k + 1}{2k - 1} \right) \ldots \left( \frac{5}{3} \right) c_1
\]

\[
= \frac{2k + 3}{3} c_1.
\]

The general solution is

\[
y = \sum_{n=0}^{\infty} c_n x^n
\]

\[
= \sum_{k=0}^{\infty} c_{2k} x^{2k} + \sum_{k=0}^{\infty} c_{2k+1} x^{2k+1}
\]

\[
= c_0 \sum_{k=0}^{\infty} (k + 1)x^{2k} + c_1 \sum_{k=0}^{\infty} \frac{2k + 3}{3} x^{2k+1}.
\]

**EXAMPLE 4** Find the general solution to $y'' - 2xy' + y = 0$.

**Solution** Assuming that

\[
y = \sum_{n=0}^{\infty} c_n x^n,
\]

substitution into the differential equation gives us

\[
\sum_{n=2}^{\infty} n(n - 1)c_n x^{n-2} - 2 \sum_{n=1}^{\infty} nc_n x^n + \sum_{n=0}^{\infty} c_n x^n = 0.
\]

We next determine the coefficients, listing them in the following table.

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<tr>
<td>$x^4$</td>
<td>$6(5)c_6 - 8c_4 + c_4 = 0$   or $c_6 = \frac{7}{6 \cdot 5} c_4$</td>
</tr>
<tr>
<td>\vdots</td>
<td>\vdots</td>
</tr>
<tr>
<td>$x^n$</td>
<td>$(n + 2)(n + 1)c_{n+2} - (2n - 1)c_n = 0$   or $c_{n+2} = \frac{2n - 1}{(n + 2)(n + 1)} c_n$</td>
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</tbody>
</table>
From the recursive relation
\[ c_{n+2} = \frac{2n - 1}{(n + 2)(n + 1)} c_n \]
we write out the first few terms of each series for the general solution:
\[
y = c_0 \left( 1 - \frac{1}{2} x^2 - \frac{3}{4!} x^4 - \frac{21}{6!} x^6 - \cdots \right)
+ c_1 \left( x + \frac{1}{3!} x^3 + \frac{5}{5!} x^5 + \frac{45}{7!} x^7 + \cdots \right).
\]

**EXERCISES 16.5**

In Exercises 1–18, use power series to find the general solution of the differential equation.

1. \[ y'' + 2y' = 0 \]
2. \[ y'' + 2y' + y = 0 \]
3. \[ y'' + 4y = 0 \]
4. \[ y'' - 3y' + 2y = 0 \]
5. \[ x^2y'' - 2xy' + 2y = 0 \]
6. \[ y'' - xy' + y = 0 \]
7. \[ (1 + x)y'' - y = 0 \]
8. \[ (1 - x^2)y'' - 4xy' + 6y = 0 \]
9. \[ (x^2 - 1)y'' + 2xy' - 2y = 0 \]
10. \[ y'' + y' - x^2y = 0 \]
11. \[ (x^2 - 1)y'' - 6y = 0 \]
12. \[ xy'' - (x + 2)y' + 2y = 0 \]
13. \[ (x^2 - 1)y'' + 4xy' + 2y = 0 \]
14. \[ y'' - 2xy' + 4y = 0 \]
15. \[ y'' - 2xy' + 3y = 0 \]
16. \[ (1 - x^2)y'' - xy' + 4y = 0 \]
17. \[ y'' - xy' + 3y = 0 \]
18. \[ x^2y'' - 4xy' + 6y = 0 \]