On the Pseudo Cross-Variogram

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Normal cross-variograms cannot be estimated from data in the usual way when there are only a few points where both variables have been measured. But the experimental pseudo cross-variogram can be computed even where there are no matching sampling points, and this appears as its principal advantage. The pseudo cross-variogram may be unbounded, though for its existence the intrinsic hypothesis alone is not a sufficient stationarity condition. In addition the differences between the two random processes must be second order stationary. Modeling the function by linear coregionalization reflects the more restrictive stationarity condition: the pseudo cross-variogram can be unbounded only if the unbounded correlation structures are the same in all variograms. As an alternative to using the pseudo cross-variogram a new method is presented that allows estimating the normal cross variogram from data where only one variable has been measured at a point.

KEY WORDS: cokriging, coregionalization, cross-correlation, temporal change, undersampling.

INTRODUCTION

Cokriging is now widely recognized as a means of estimating values of spatially cross-correlated variables at unsampled points and over larger blocks. It can be used to estimate all variables in a set, either to improve precision (Journel and Huijbregts, 1978; Wackernagel, 1993) or for coherence (Matheron, 1979; Myers, 1983). However, its principal use has been to estimate values of a sparsely sampled variable from data on it plus those of one or more variables that have been sampled more densely, commonly termed "the undersampled case". In these circumstances it can confer substantial benefits in terms of precision, efficiency, and cost (Atkinson et al., 1992).

The cokriging equations are usually written in terms of variograms. These must be estimated from data, and for each experimental normal cross-variogram there must be numerous places at which values of both variables are known. In the undersampled case only a subset of the values of the fully sampled variable

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is used to estimate the cross-variogram: the information contained in the remainder is wasted. The undersampled variable might have been measured at points where the other was not, and in the extreme there might be no place at which both were measured. In the latter case it is impossible to compute an experimental normal cross-variogram in the usual way. Where the variables can be modeled as second order stationary random processes the covariances can replace the semivariances in the kriging equations. This avoids the above difficulty, since cross-covariances can be estimated regardless of the sampling configuration.

However, variables often appear to have unbounded variation, and resort must be made to the variograms to express the spatial correlation. This line of reasoning led Clark et al. (1989) to introduce the pseudo cross-variogram for the purpose. They defined it as the expected squared difference between two random variables at two different points (see below); so it could be unbounded and they could estimate it without having data on both variables at the same places. Myers (1991) investigated the function, showed how it could be used for cokriging, and generally confirmed its advantages.

We encounter the extreme case of undersampling when we study temporal change in the soil. Our variables are in fact the values of the same property at different times, and we treat them as realizations of cross-correlated random processes (Papritz and Flühler, 1993). Our sampling is destructive. At each sampling point we take the soil away to the laboratory for measurement, it is not there on subsequent occasions, and the surroundings of the point may be grossly disturbed. Thus we cannot measure any two variables at identical points. We have therefore been attracted to the pseudo cross-variogram, and we have examined it for cokriging change with time. In the process we have discovered that it demands more severe assumptions of stationarity than originally supposed. Still, it has some advantages compared to the cross-covariance function. In this paper we explore the fundamental characteristics of the function.

**PROPERTIES OF THE PSEUDO CROSS-VARIOGRAM**

Consider two intrinsic random processes of order zero, $Z_1(x)$ and $Z_2(x)$, with means $\mu_1$ and $\mu_2$, with variograms $\gamma_{11}(h)$ and $\gamma_{22}(h)$, and with a normal cross-variogram defined by

$$\gamma_{21}(h) = \frac{1}{2}E\{Z_2(x + h) - Z_2(x)\}$$ \(1\)

where in the usual notation $h$ is a vector separating two places, $x$ and $x + h$. For this situation Clark et al. (1989) defined the pseudo cross-variogram by

$$\gamma_{21}^c(h) = \frac{1}{2}E\{|Z_2(x + h) - Z_1(x)\}^2\]$$ \(2\)

Myers (1991) observed that the quantities, $\gamma_{21}^c(h)$, defined in this way are equal
to the variances of the differences only if both $Z_1(x)$ and $Z_2(x)$ have constant and equal means ($\mu_1 = \mu_2$). He therefore proposed the more general definition for the pseudo cross-variogram $\gamma_{21}^p(h)$ as half the variance of the differences:

$$\gamma_{21}^p(h) = \frac{1}{2} \text{Var} [Z_2(x + h) - Z_1(x)]$$  \hspace{1cm} (3)

If the means differ then the function defined by Clark et al. equals $\gamma_{21}^p + (\mu_1 - \mu_2)^2$.

Unlike the normal cross-variogram the pseudo cross-variogram is not an even function, i.e., $\gamma_{21}^p(h)$ in general does not equal $\gamma_{21}^p(-h)$ and it may differ from $\gamma_{12}^p(h)$. The only identity that holds is

$$\gamma_{21}^p(h) = \gamma_{12}^p(-h)$$  \hspace{1cm} (4)

If $Z_1(x)$ and $Z_2(x)$ are second order stationary with covariance functions $C_{11}(h)$ and $C_{22}(h)$ and with the cross-covariance function $C_{21}(h)$ then the pseudo cross-variogram is related to these by

$$\gamma_{21}^p(h) = \frac{1}{2} C_{11}(0) - C_{21}(h)$$  \hspace{1cm} (5)

Thus, since $C_{21}(\infty) = 0$ for bounded variograms, the sill of the pseudo cross-variogram equals the average of the sills of the two autovariograms, $\gamma_{11}(h)$ and $\gamma_{22}(h)$. Equation (5) further shows that the pseudo cross-variogram can describe negative correlation between second order stationary processes. If the cross-covariance function is negative and monotonically increases to zero for increasing lag distance then the pseudo cross-variogram is greatest at lag zero and decreases to its “sill”. The pseudo cross-variogram, consisting of variances, is always larger than or equal to zero. If $Z_1(x)$ and $Z_2(x)$ are perfectly and positively correlated for lag zero, i.e., if $C_{21}(0) = \{C_{11}(0)C_{22}(0)\}^{1/2}$, and if their variances are equal, $C_{11}(0) = C_{22}(0)$, then the pseudo cross-variogram will be zero for lag zero. In all other cases $\gamma_{21}^p(h)$ will be positive.

With the autovariograms and the pseudo cross-variogram we can express the covariance between two arbitrary differences in their most general form as

$$\text{Cov} [Z_i(x) - Z_j(y), Z_k(u) - Z_l(v)] = -\gamma_{ik}^p(x - u) + \gamma_{ij}^p(x - y) + \gamma_{jk}^p(y - u) - \gamma_{jl}^p(y - v)$$  \hspace{1cm} (6)

where $i, j, k,$ and $l$ refer to four variables and $x, y, u,$ and $v$ refer to four places. If two indices coincide then we have a semivariance and the superscript $p$ is redundant. In the special case in which $v = y$ and $j = l$ this is identical to formula (5') in Myers (1991). To prove Eq. (6) we write the right-hand side as

$$\text{Cov} [Z_i(x) - Z_l(v) + Z_l(v) - Z_j(y), Z_k(u) - Z_l(v)] = \text{Cov} [Z_i(x) - Z_l(v), Z_k(u) - Z_l(v)] + \text{Cov} [Z_l(v) - Z_j(y), Z_k(u) - Z_l(v)]$$
Then Eq. (6) follows directly from Eq. (5') of Myers (1991). In particular, we obtain from Eq. (6)

\[
\gamma_{21}(h) = \frac{1}{2} \gamma_{21}^p(h) + \frac{1}{2} \gamma_{12}^p(h) - \gamma_{21}^p(0)
\]  

(7)

Hence the usual cross-variogram can be expressed in terms of the pseudo cross-variogram. The converse is not possible: we cannot obtain the pseudo cross-variogram from the cross-variogram unless the pseudo cross-variogram is even and more restrictive stationarity conditions are satisfied. When \( Z_1(x) \) and \( Z_2(x) \) are merely intrinsic of order zero the pseudo cross-variogram does not in general exist. Equation (7) suggests that we need in addition that the variance of \( Z_2(x) - Z_1(x) \) is finite and independent of \( x \). On the other hand there are situations where \( Z_1(x) \) and \( Z_2(x) \) are not stationary, but the pseudo cross-variogram exists nevertheless. The following example shows what can happen.

Let \( Z_1(x) \) and \( \epsilon(x) \) be two uncorrelated intrinsic processes with mean zero. From these we form \( Z_2(x) \) as the following linear combination

\[
Z_2(x) = \sum_{j=1}^{p} \lambda_j Z_1(x - h_j) + \epsilon(x)
\]

with fixed coefficients, \( \lambda_j \), and fixed lags, \( h_j \). Then \( Z_2(x) \) is intrinsic with \( E[Z_2(x)] = 0 \),

\[
\gamma_{22}(h) = \sum_{j=1}^{p} \sum_{k=1}^{p} \lambda_j \lambda_k [\gamma_{11}(h - h_j + h_k) - \gamma_{11}(h_j - h_k)] + \gamma_{\epsilon\epsilon}(h)
\]

(9)

and

\[
\gamma_{21}(h) = \sum_{j=1}^{p} \lambda_j [\frac{1}{2} \gamma_{11}(h - h_j) + \frac{1}{2} \gamma_{11}(h_j + h_k) - \gamma_{11}(h_j)]
\]

(10)

If we want the pseudo cross-variogram to exist then in addition \( \sum_{j=1}^{p} \lambda_j \) must equal 1, and \( \epsilon(x) \) must be second order stationary. Then we find

\[
\gamma_{21}^p(h) = \frac{1}{2} C_{\epsilon\epsilon}(0) + \sum_{j=1}^{p} \lambda_j \gamma_{11}(h - h_j) - \frac{1}{2} \sum_{j=1}^{p} \sum_{k=1}^{p} \lambda_j \lambda_k \gamma_{11}(h_j - h_k)
\]

(11)

Note that under these conditions \( Z_1(x) \) and \( Z_2(x) \) are in general only intrinsic, but not stationary. Both \( Z_1(x) \) and \( Z_2(x) \) can fluctuate substantially, but their fluctuations are mainly parallel so that the differences, \( Z_1(x) - Z_2(x) \), fluctuate much less.

This example can be generalized in two ways. First, we can introduce dependence between \( Z_1(x) \) and \( \epsilon(x) \). For the existence of \( \gamma_{22} \) and \( \gamma_{21} \) it is sufficient that the cross-variogram \( \gamma_{\epsilon1} \) exists. For the existence of \( \gamma_{21}^p \) we need \( Z_1(x + h) - Z_1(x) \) and \( \epsilon(x) \) to be jointly second order stationary for any \( h \).
This is again more restrictive. Second, another generalization is to replace the sum in Eq. (8) by an integral

\[ Z_2(x) = \int \lambda(y) \, Z_1(x - y) \, dy + \epsilon(x) \]  

(12)

This can be obtained by letting \( p \) in Eq. (8) tend to infinity. For a rigorous treatment we need \( \int |\lambda(x)| \, dx < \infty \). Formulae similar to (9) to (11) hold.

Equations (9) to (11) show that the slopes of \( \gamma_{22}(h) \), \( \gamma_{21}(h) \), and \( \gamma_{21}^p(h) \) are related to the slope of \( \gamma_{11}(h) \). Indeed, if the pseudo cross-variogram exists then the slope of the variogram \( \gamma_{22}(h) \) cannot be independent of \( \gamma_{11}(h) \). Under general conditions it can be shown (see Appendix) that in the limiting situation where the lag distance increases to infinity the slopes of \( \gamma_{22}(h) \), \( \gamma_{21}(h) \), \( \gamma_{21}^p(h) \), and \( \gamma_{11}(h) \) must all be equal. This defines a criterion for determining whether the pseudo cross-variogram of two intrinsic random processes exists. If their variograms increase for large lag distances at the same rate then the necessary stationarity conditions for the pseudo cross-variogram are likely to be satisfied. If on the other hand the slopes of the variograms are different then \( \gamma_{21}^p(h) \) does not exist. There are no similar restrictions on the variograms \( \gamma_{11}(h) \) and \( \gamma_{22}(h) \) if \( \epsilon(x) \) satisfies only the intrinsic hypothesis. Of course, the normal cross-variogram must satisfy Schwarz's inequality for all \( h \):

\[ |\gamma_{21}(h)| \leq \sqrt{\gamma_{11}(h) \, \gamma_{22}(h)} \]  

(13)

Equation (13) can be used to derive conditions on the slope of \( \gamma_{21}(h) \), but apart from these no other restrictions on the slope of the variograms exist.

**ESTIMATING AND MODELING**

A possible estimator to compute the experimental pseudo cross-variogram from sample data is

\[ \hat{\gamma}_{21}^p(h) = \frac{1}{2N(h)} \sum_{i=1}^{N(h)} \left[ (Z_2(x_i + h) - \bar{Z}_2) - (Z_1(x_i) - \bar{Z}_1) \right]^2 \]  

(14)

where \( \bar{Z}_1 \) and \( \bar{Z}_2 \) denote the arithmetic means of \( n_1 \) and \( n_2 \) observations. This estimator is biased. For small lag distances the bias equals approximately the variance of the difference of the sample means (see Priestley, 1981).

\[ \text{Bias } [\hat{\gamma}_{21}^p(h)] \approx \text{Var } [\bar{Z}_2 - \bar{Z}_1] = \frac{2}{n_1n_2} \sum_{i=1}^{n_1} \sum_{j=1}^{n_2} \gamma_{21}^p(x_j - x_i) \]

\[ - \frac{1}{n_1^2} \sum_{i=1}^{n_1} \sum_{j=1}^{n_1} \gamma_{11}(x_j - x_i) - \frac{1}{n_2^2} \sum_{i=1}^{n_2} \sum_{j=1}^{n_2} \gamma_{22}(x_j - x_i) \]  

(15)

Myers (1991) has suggested a different estimator for the experimental pseudo
cross-variogram, namely the average non-centered squared differences of the variables. We think that this estimator should be considered only where \( \mu_1 \) and \( \mu_2 \) are known to be equal. This may be so where we regard \( Z_1(x) \) and \( Z_2(x) \) as deterministic mean functions plus random residuals, only the correlation structure of which has to be analyzed. In general, the observations should be centered before computing the squared differences. Further, suitable physical units should be chosen to avoid numerical problems. The deviations from the means should be of the same magnitude for both variables. This amounts to standardizing the variables by some constants. This is also favorable for comparing the variograms of two variables with vastly different scale. If both the variograms and the pseudo cross-variogram are estimated and modeled from the standardized data then the kriging estimates will not be affected by the standardization, and the estimation variances can easily be transformed back into the unit of choice.

Models must be fitted to experimental variograms or covariance functions for kriging, and to ensure that the kriging system has a unique solution either the normal cross-variogram or the pseudo cross-variogram must be modeled with the autovariograms. The basic condition is that the variogram matrix \( \Gamma(h) \):

\[
\Gamma(h) = \begin{bmatrix}
\gamma_{11}(h) & \gamma_{12}^p(h) \\
\gamma_{21}^p(h) & \gamma_{22}(h)
\end{bmatrix}
\]  

(16)

must be conditionally negative semi-definite (CNSD).

In principle, \( \gamma_{11}(h) \), \( \gamma_{22}(h) \), and \( \gamma_{21}^p(h) \) could be modeled to satisfy Eqs. (8), (9), and (11). Usually, however, too little is known to use such an approach, and multivariate spatial random processes are therefore modeled as linear co-regionalizations (Journel and Huijbregts, 1978; Wackernagel, 1988; Goulard and Voltz, 1992). One decomposes a process \( Z_k(x) \) into \( S \) orthogonal random processes \( Z^*_k(x) \) (\( s = 1, \ldots, S \)), each having a distinct autocorrelation structure and that together sum to \( Z_k(x) \). Each component \( Z^*_k(x) \) is in turn modeled as a linear combination of \( P \) mutually uncorrelated but spatially autocorrelated elementary random processes, \( Y^*_p(x) \), all having a common elementary variogram \( g^*(h) \). Formally, \( Z_k(x) \) can then be written

\[
Z_k(x) = \sum_{s=1}^{S} Z^*_k(x) + \mu_k = \sum_{s=1}^{S} \sum_{p=1}^{P} a_{kp}^s Y^*_p(x) + \mu_k
\]  

(17)

where \( \mu_k \) is the mean of the \( k \) th process. If one denotes the sum \( \Sigma_{p=1}^{P} a_{kp}^s a_{kp}^l \) by \( b_{jk}^s \) then the autovariograms and the normal cross-variogram can be represented by

\[
\gamma_{jk}(h) = \sum_{s=1}^{S} b_{jk}^s g^*(h)
\]  

(18)
A sufficient condition for the variogram matrix \( \Gamma(h) \) to be CNSD is that the matrices of coefficients \([b_j]\) are positive semi-definite.

The pseudo cross-variogram exists only if for each \( s \) either \( a^s = a^s = a^s \), or \( Y_s(x) \) is second order stationary with covariance function \( C^s(h) \). Note that the first condition implies \( b_{11}^s = b_{22}^s = b_{21}^s = b^s \). Without loss of generality we may assume that the first condition holds for \( s \leq S_o \) the number of unbounded correlation structures, and the second for \( s > S_o \). Then we obtain

\[
\gamma_{jk}(h) = \sum_{s=1}^{S_o} b^s g^s_{jk}(h) + \sum_{s=S_o+1}^S b^s_{jk}[C^s(0) - C^s(h)]
\]

and

\[
\gamma_{21}(h) = \sum_{s=1}^{S_o} b^s g^s_{21}(h) + \sum_{s=S_o+1}^S \left( \frac{1}{2}(b_{11}^s + b_{22}^s)C^s(0) - b_{21}^sC^s(h) \right)
\]

Two remarks must be added. First, Eqs. (19) and (20) once again confirm the additional stationarity required for the existence of the pseudo cross-variogram. The unbounded components must be the same for all variograms; only the bounded components may differ. This is one step away from the assumption of second order stationarity, though we still lack experience to say how useful this extension is likely to be in practice with extreme undersampling.

Second, an asymmetric pseudo cross-variogram cannot be modeled using Eq. (20) because the elementary variograms, \( g^s(h) \), and covariance functions, \( C^s(h) \), are even. If the asymmetry consists simply of a "lag effect", i.e., if the pseudo cross-variogram can be made symmetrical by shifting one variable with respect to the other by some vector \( h \neq 0 \) then the linear model may be generalized (Journel and Huijbregts, 1978, p. 173) to describe such asymmetry. Other types of asymmetries, however, cannot be modeled by linear coregionalization. Modeling these requires more detailed assumptions about the structure of the coregionalization. Equations (8) and (11) are an example of such a model. But unlike multivariate temporal phenomena, asymmetrical cross-correlation patterns are not likely to occur frequently in space. Thus, the limitations inherent in the linear model of coregionalization are not seriously restricting.

A special problem arises in modeling the pseudo cross-variogram where the two variables are sampled at completely different places and there is a nugget variance, i.e., where for some \( s \) \( g^s(0) = 0 \) and \( g^s(h) = 1 \) for \( |h| > 0 \). The experimental pseudo cross-semivariance \( \hat{\gamma}_{21}(0) \) cannot be computed if no sampling locations coincide. Equation (20) shows, however, that \( b_{21}^s \) can be determined only if the pseudo cross-semivariance for \( |h| = 0 \) is known. Thus, the exact value of \( b_{21}^s \) remains unknown here, but its allowed bounds are obtained from Schwarz's inequality: \( |b_{21}^s| \) must lie in the interval \([0, \sqrt{b_{11}^s b_{22}^s}]\). Estimating and modeling the cross-covariance function from data from different
locations suffers from the same limitation. As with the pseudo cross-variogram, its nugget constant cannot be determined. The uncertainty about the nugget constant affects the prediction by cokriging if either the term \( C_{21}(0) \) or \( \gamma_{21}(0) \) appears in the cokriging system.

**ALTERNATIVE FOR ESTIMATING THE NORMAL CROSS-VARIOGRAM**

We have shown above that using the pseudo cross-variogram does not completely circumvent the problems that arise from strong undersampling and unbounded variation. Therefore, the following alternative for estimating and modeling the normal cross-variogram may be helpful where no or only a few sampling locations of the different variables match. Assume that \( Z_1(x) \) is observed at locations \( x_{1,i}, i = 1, \ldots, n_1 \), and that \( Z_2(x) \) is observed at locations \( x_{2,k}, k = 1, \ldots, n_2 \), with \( x_{1,i} \neq x_{2,k} \forall (i, k) \). Then form linear combinations of the observed values such that for each variable the weights sum to zero, and use the identity:

\[
E\left[\{Z_1(x) - Z_1(u)\} \{Z_2(y) - Z_2(u)\}\right] = \gamma_{12}(x - u) + \gamma_{12}(y - u) - \gamma_{12}(x - y) \quad (21)
\]

From this one finds

\[
E\left[\sum_{i=1}^{n_1} \lambda_i Z_1(x_{1,i}) \sum_{k=1}^{n_2} \lambda_k Z_2(x_{2,k})\right] = -\sum_{i=1}^{n_1} \sum_{k=1}^{n_2} \lambda_i \lambda_k \gamma_{12}(x_{1,i} - x_{2,k}) \quad (22)
\]

where \( \sum_{i=1}^{n_1} \lambda_i = \sum_{k=1}^{n_2} \lambda_k = 0 \). Note that the identity given in Eq. (21) is a prerequisite for expressing a cokriging system in terms of the variograms (Myers, 1982, Eq. 16) and adds no further restrictions on the properties of the random processes.

Equation (22) suggests a method for estimating and modeling the cross-variogram if the sampling locations of the two processes do not coincide. The sum of squared deviations \( Q_{12} \) as defined in Eq. (23) must then be minimized by some nonlinear least square fitting scheme to determine the vector of model parameters \( \theta_{12} \).

\[
Q_{12} = \sum_{\lambda_1, \lambda_2} \left[ \sum_{i=1}^{n_1} \lambda_i^1 Z_1(x_{1,i}) \sum_{k=1}^{n_2} \lambda_k^2 Z_2(x_{2,k}) \right. \\
\left. + \sum_{i=1}^{n_1} \sum_{k=1}^{n_2} \lambda_i^1 \lambda_k^2 \gamma_{12}(x_{1,i} - x_{2,k}; \theta_{12}) \right]^2 \quad (23)
\]

The first summation operator denotes summation over all possible sets of weights.
that can be sensibly chosen. One obvious choice of weights for the first variable is \( \lambda_i = 1, \lambda_j = -1, i \neq j \) and \( \lambda_k = 0, \forall k \neq (i, j) \). The weights for the second variable may be chosen similarly: \( \lambda_k = 1, \lambda_l = -1, k \neq l \) and \( \lambda_l = 0, \forall l \neq (k, l) \). The sum \( Q_{12} \) can then be written

\[
Q_{12} = \sum_{i=1}^{n_1} \sum_{j=i+1}^{n_2} \sum_{k=1}^{n_2} \sum_{l=k+1}^{n_2} \left\{ [Z_1(x_{1,i}) - Z_1(x_{1,j})] [Z_2(x_{2,k}) - Z_2(x_{2,l})] \\
+ \gamma_{12}(x_{1,i} - x_{2,k}; \theta_{12}) - \gamma_{12}(x_{1,i} - x_{2,l}; \theta_{12}) \\
- \gamma_{12}(x_{1,j} - x_{2,k}; \theta_{12}) + \gamma_{12}(x_{1,j} - x_{2,l}; \theta_{12}) \right\}^2
\] (24)

Thus, for this choice of weights the four variogram terms must be fitted to the cloud of \((Z_1^i Z_2^j)\) \((Z_1^i Z_2^j)\) products of differences. This procedure is closely related to fitting directly the variogram cloud. The autovarioograms, \( \gamma_{11}(h) \) and \( \gamma_{22}(h) \), can be modeled correspondingly. The sum of squared deviations, \( Q_{11} \) and \( Q_{22} \), are given by

\[
Q_{kk} = \sum_{i=1}^{n_k-1} \sum_{j=i+1}^{n_k} \left\{ [Z_k(x_{k,i}) - Z_k(x_{k,j})]^2 - \gamma_{kk}(x_{k,i} - x_{k,j}; \theta_{kk}) \right\}^2
\] (25)

A bivariate linear model of coregionalization can then be fitted by minimizing the sum \( Q = w_{11}Q_{11} + w_{22}Q_{22} + w_{12}Q_{12} \), where \( w_{jk} \) are weights used to control the influence of the different variogram clouds. To ensure that the variogram matrix \( \Gamma(h) \) is CNSD the fitting algorithm must check that the matrices of coefficients, \( [b_{jk}] \), are positive semi-definite.

Although similar techniques are used to estimate and to model generalized covariances (Delfiner, 1976), the approach might give rise to practical difficulties.

First, the fitting algorithm soon demands too much computing time if samples are large. For example, if one has samples of size \( n_1 = n_2 = 100 \) then from Eq. (24) it can be seen that approximately \( 10^8 \) values of the cross-variogram are needed to compute \( Q_{12} \). Since the models are nonlinear in the scale parameters fitting must be iterative, and the number of operations is a multiple of \( 10^8 \). Currently, we are developing and exploring computationally feasible approximations for minimizing \( Q_{12} \) and \( Q \), respectively.

Second, robust fitting techniques might be necessary to reduce the influence of extreme values in the variogram cloud. Despite these difficulties the new approach is promising because it uses all the information on the fully sampled variable to establish the cross-correlation with the undersampled variable. Further, it circumvents the limitations inherent in attempting to cokrige from the pseudo cross-variogram or the cross-covariance function.
CONCLUSION

The pseudo cross-variogram is undoubtedly attractive for describing cross-correlation between two variables where measurements on both have not been or cannot be made at the same places and where there is severe undersampling. However, its main application in practice is likely to be for cokriging. Although it may be unbounded in theory the stationarity condition required for its existence is more restricting than the intrinsic hypothesis. Modeling it by linear coregionalization either restricts it to second order stationary processes, or the unbounded structures of the auto- and the pseudo cross-variogram must be the same. Thus, it offers some advantage over the cross-covariance function which can be used only for second order stationary processes.

Another way to overcome the difficulties arising with unbounded variation and strong undersampling is the proposed method for estimating and modeling the normal cross-variogram. Further investigations are necessary to decide which approach is more advantageous.

APPENDIX

Relation Between the Slopes of the Auto- and Pseudo Cross-Variograms

Using the definition of the pseudo cross-variogram (Eq. 3) and expanding the term $\text{Var}[Z_2(x+h) - Z_1(x+h) + Z_1(x+h) - Z_1(x)]$, one finds

$$\gamma^p_{21}(h) = \gamma^p_{21}(0) + \gamma_{11}(h) + \text{Cov}[Z_2(x+h) - Z_1(x+h), Z_1(x+h) - Z_1(x)]$$

(26)

We are interested in the relation between $\gamma^p_{21}(h)$ and $\gamma_{11}(h)$ in the limit $h \to \infty$ where the influence of the second order stationary component of $Z_2(x)$ has decayed to zero. We can divide Eq. (26) by $\gamma_{11}(h)$ and form the limit

$$\lim_{h \to \infty} \frac{\gamma^p_{21}(h)}{\gamma_{11}(h)} = \lim_{h \to \infty} \frac{\gamma^p_{21}(0)}{\gamma_{11}(h)} + \lim_{h \to \infty} \frac{\gamma_{11}(h)}{\gamma_{11}(h)}$$

$$+ \lim_{h \to \infty} \frac{\text{Cov}[Z_2(x+h) - Z_1(x+h), Z_1(x+h) - Z_1(x)]}{\gamma_{11}(h)}$$

(27)

For an unbounded variogram $\gamma_{11}(h)$, evidently

$$\lim_{h \to \infty} \frac{\gamma^p_{21}(0)}{\gamma_{11}(h)} = 0$$

(28)

and

$$\lim_{h \to \infty} \frac{\gamma_{11}(h)}{\gamma_{11}(h)} = 1$$

(29)
As a result of Schwarz's inequality, which states that

$$0 \leq |\text{Cov}[Z_2(x + h) - Z_1(x + h), Z_1(x + h) - Z_1(x)]|$$

$$\leq \sqrt{\text{Var}[Z_2(x + h) - Z_1(x + h)] \text{Var}[Z_1(x + h) - Z_1(x)]}$$

$$= 2 \sqrt{\gamma_{21}^p(0) \gamma_{11}(h)}$$

(30)

the third term of the right-hand side of Eq. (27) becomes

$$0 \leq \lim_{h \to \infty} \frac{|\text{Cov}[Z_2(x + h) - Z_1(x + h), Z_1(x + h) - Z_1(x)]|}{\gamma_{11}(h)}$$

$$\leq \lim_{h \to \infty} \frac{2 \sqrt{\gamma_{21}^p(0) \gamma_{11}(h)}}{\gamma_{11}(h)} = \lim_{h \to \infty} \frac{2 \sqrt{\gamma_{21}^p(0)}}{\sqrt{\gamma_{11}(h)}} = 0$$

(31)

Thus, it follows that

$$\lim_{h \to \infty} \frac{\gamma_{21}^p(h)}{\gamma_{11}(h)} = 1$$

(32)

which in turn implies that the slopes of $\gamma_{21}^p(h)$ and $\gamma_{11}(h)$ must be equal for large lag distances.

By analogous argument, i.e., by expanding $\text{Var}[Z_2(x + h) - Z_2(x) + Z_2(x) - Z_1(x)]$, it can be shown that

$$\lim_{h \to \infty} \frac{\gamma_{21}(h)}{\gamma_{22}(h)} = 1$$

(33)

Furthermore from Eq. (7) it follows immediately that

$$\lim_{h \to \infty} \frac{\gamma_{21}(h)}{\gamma_{21}^p(h)} = 1$$

(34)

And so one can conclude that

$$\lim_{h \to \infty} \frac{\gamma_{21}(h)}{\gamma_{21}(h)} = 1$$

(35)

Thus, the variograms $\gamma_{11}(h)$, $\gamma_{22}(h)$, $\gamma_{21}(h)$, and $\gamma_{21}^p(h)$ must all have the same slopes for large lag distances.

REFERENCES


