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APPLICATIONS OF MULTIDIMENSIONAL FIXED-POINT THEOREMS TO A NONLINEAR INTEGRAL EQUATION

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Abstract: The purpose of this paper is to present the applications of multidimensional fixed point theorems. For this, we provide a multidimensional fixed point theorem and then using this theorem we prove the existence and uniqueness of solution of nonlinear integral equation.

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Key Words: fixed-point, partially ordered metric space, integral equation

1. Introduction and Preliminaries

The aim of this paper is to prove the existence and uniqueness of solution of nonlinear integral equation. For this, we use the methods of multidimensional fixed point theorems. The concept of multidimensional fixed point i.e., Υ-*fixed point* was introduced by Roldàn *et. al.* [10, 11]. This notion covers the concepts of *coupled, tripled, quadruple* fixed point. We refer the reader to the references [6, 1, 4, 5, 9, 12, 2, 3, 7, 8] in which were introduced the concept of

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coupled, tripled, quadruple fixed points and obtained related theorems. The coupled, tripled, quadruple and Υ-fixed point theorems have wide potential application in various branches of mathematics, such as differential equations, mathematical economics, game theory, dynamics, optimal control, functional analysis, operator theory etc. In this work we focus to the applications of Υ-fixed point theorem to to a nonlinear integral equation. Let us recall the necessary notions a definition of Υ-fixed point. Denote by $(X, d, \preceq)$ a *partially ordered metric space*.

Definition 1.1. An ordered metric space $(X, d, \preceq)$ is called *regular* if it satisfies the following:

- if $\{x_m\}$ is a nondecreasing sequence and $\{x_m\} \overset{d}{\to} x$, then $x_m \preceq x$ for all m;

- if $\{y_m\}$ is a nonincreasing sequence and $\{y_m\} \overset{d}{\to} y$,Abstract and Applied Analysis Vol. (2012), 134701, 1-12, 2012

  then $y_m \succeq y$ for all m.

Taking a natural number $k \geq 2$ we consider the set $\Lambda_k = \{1, 2, \ldots, k\}$. Let $\{\mathcal{A}, \mathcal{B}\}$ be a partition of Λ_k that is $\mathcal{A} \cup \mathcal{B} = \Lambda_k$ and $\mathcal{A} \cap \mathcal{B} = \emptyset$. Using this partition and partially ordered metric space $(X, d, \preceq)$ we define a k-dimensional partially ordered metric space $(X^k, \mathbf{d}_k, \preceq_k)$ as follows:

- the k-cartesian power of a set X

  $$X^k = \underbrace{X \times X \times \cdots \times X}_{k} = \{(\mathbf{x} = (x_1, x_2, ..., x_k)) : |x_i \in X \text{ for all } i \in \Lambda_k\};$$

- the maximum metric $\mathbf{d}_k : X^k \times X^k \to [0, +\infty)$, given by

  $$\mathbf{d}_k(\mathbf{x}, \mathbf{y}) = \max_{1 \leq i \leq k} \{d(x_i, y_i)\},$$

  where $\mathbf{x} = (x_1, x_2, ..., x_k), \mathbf{y} = (y_1, y_2, ..., y_k) \in X^k$;

- the partial order w.r.t $\{\mathcal{A}, \mathcal{B}\}$ that is, for any $\mathbf{x} = (x_1, x_2, ..., x_k)$ and $\mathbf{y} = (y_1, y_2, ..., y_k) \in X^k$ we have

  $$\mathbf{x} \preceq_k \mathbf{y} \Leftrightarrow \begin{cases} x_i \preceq y_i, & \text{if } \quad i \in \mathcal{A}, \\ x_i \succeq y_i, & \text{if } \quad i \in \mathcal{B}. \end{cases}$$

It is easy to see that if (X, d) is a complete metric space, then $(X^k, \mathbf{d}_k)$ is a complete metric space.

Definition 1.2. We say that a mapping $F : X^k \to X$ has the *mixed monotone* property w.r.t partition $\{\mathcal{A}, \mathcal{B}\}$, if F is monotone nondecreasing in arguments of $\mathcal{A}$ and monotone nonincreasing in arguments of $\mathcal{B}$.

We define the following set of mappings:

$$\Omega_{\mathcal{A},\mathcal{B}} = \{\sigma : \Lambda_k \to \Lambda_k : \sigma(\mathcal{A}) \subseteq \mathcal{A},\ \sigma(\mathcal{B}) \subseteq \mathcal{B}\},$$

$$\Omega'_{\mathcal{A},\mathcal{B}} = \{\sigma : \Lambda_k \to \Lambda_k : \sigma(\mathcal{A}) \subseteq \mathcal{B},\ \sigma(\mathcal{B}) \subseteq \mathcal{A}\}.$$

Let $\Upsilon = (\sigma_1, \sigma_2, \ldots, \sigma_k)$ be k-tuple of mappings of $\sigma_i : \Lambda_k \to \Lambda_k$ such that $\sigma_i \in \Omega_{\mathcal{A},\mathcal{B}}$ if $i \in \mathcal{A}$ and $\sigma_i \in \Omega'_{\mathcal{A},\mathcal{B}}$ if $i \in \mathcal{B}$. In the sequel we consider only such kind of k-tuple of mappings.

Definition 1.3. A point $\mathbf{x} = (x_1, x_2, \ldots, x_k) \in X^k$ is called Υ-*fixed point* of a mapping $F : X^k \to X$ if

$$F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, \ldots, x_{\sigma_i(k)}) = x_i$$

for all $i \in \Lambda_k$.

2. Multidimensional Fixed Point Theorems

In this section we provide relations between one and multidimensional fixed point theorems. Define $T_\Upsilon : X^k \to X^k$ as follows:

$$T_\Upsilon(x_1, x_2, \ldots, x_k) = \Big(F(x_{\sigma_1(1)}, x_{\sigma_1(2)}, \ldots, x_{\sigma_1(k)}), F(x_{\sigma_2(1)}, x_{\sigma_2(2)}, \ldots, x_{\sigma_2(k)}) \\ \ldots, F(x_{\sigma_k(1)}, x_{\sigma_k(2)}, \ldots, x_{\sigma_k(k)})\Big)$$

for all $\mathbf{x} = (x_1, x_2, \ldots, x_k) \in X^k$. The following theorem was proved by Roldàn.

Theorem 2.1. *[11] Let $(X, d, \preceq)$ be a complete partially ordered metric space. Let $\Upsilon : \Lambda_k \to \Lambda_k$ be a k-tuple of mappings $\Upsilon = (\sigma_1, \sigma_2, ..., \sigma_k)$ which is verifying $\sigma_i \in \Omega_{\mathcal{A},\mathcal{B}}$ if $i \in \mathcal{A}$ and $\sigma_i \in \Omega'_{\mathcal{A},\mathcal{B}}$ if $i \in \mathcal{B}$.*

- *If F has the mixed monotone property, then T_Υ is monotone nondecreasing w.r.t $\preceq_k$.*
- *If F is continuous, then T_Υ is also continuous.*
- *A point $\mathbf{x} = (x_1, x_2, \ldots, x_k) \in X^k$ is a Υ-fixed point of F, if and only if $\mathbf{x} = (x_1, x_2, \ldots, x_k)$ is a fixed point of T_Υ.*

We need the following definition which was introduced by Khan *et. al.* in [**?**].

Definition 2.2. A function $\psi : [0,+\infty) \to [0,+\infty)$ is called *altering distance function*, if ψ is continuous, monotonically increasing and $\psi(\{0\}) = \{0\}$.

Theorem 2.3. *[11] Let $(X, d, \preceq)$ be a complete partially ordered metric space. Let $\Upsilon : \Lambda_k \to \Lambda_k$ be a k-tuple of mappings $\Upsilon = (\sigma_1, \sigma_2, ..., \sigma_k)$ which is verifying $\sigma_i \in \Omega_{\mathcal{A},\mathcal{B}}$ if $i \in \mathcal{A}$ and $\sigma_i \in \Omega'_{\mathcal{A},\mathcal{B}}$ if $i \in \mathcal{B}$. Suppose $F : X^k \to X$ satisfies the following conditions:*

(i) there exist altering distance functions ψ, φ such that for all $\mathbf{x} = (x_1, x_2, \ldots, x_k)$, $\mathbf{y} = (y_1, y_2, \ldots, y_k) \in X^k$ with $\mathbf{x} \preceq_k \mathbf{y}$

$$\psi(d(F(x_1, x_2, \ldots, x_k), F(y_1, y_2, \ldots, y_k))) \leq \psi(\mathbf{d}_k(\mathbf{x}, \mathbf{y})) - \varphi(\mathbf{d}_k(\mathbf{x}, \mathbf{y}));$$

(ii) there exists $\mathbf{x}^0 = (x_1^0, x_2^0, \ldots, x_k^0) \in X^k$ such that

$$x_i^0 \preceq F(x^0_{\sigma_i(1)}, x^0_{\sigma_i(2)}, \ldots, x^0_{\sigma_i(k)})$$

if $i \in \mathcal{A}$ and $x_i^0 \succeq F(x^0_{\sigma_i(1)}, x^0_{\sigma_i(2)}, \ldots, x^0_{\sigma_i(k)})$ if $i \in \mathcal{B}$;

(iii) F has the mixed monotone property w.r.t $\{\mathcal{A}, \mathcal{B}\}$;

(iv) For all $i \in \Lambda_k$, the mapping σ_i is a permutation of Λ_k;

(v) (a) F is continuous or
(b) $(X, d, \preceq)$ is regular.

Then F has at least one Υ-fixed point.

Note that, in this theorem, the uniqueness of Υ-fixed point can easily be proven under the following additional condition.

Remark 2.4. *If for any $\mathbf{x} = (x_1, x_2, \ldots, x_k)$, $\mathbf{y} = (y_1, y_2, \ldots, y_k) \in X^k$ there exists a $\mathbf{z} = (z_1, z_2, \ldots, z_k) \in X^k$, such that $\mathbf{x} \preceq_k \mathbf{z}$ and $\mathbf{y} \preceq_k \mathbf{z}$ $\mathbf{x}^* = (x_1^*, x_2^*, \ldots, x_k^*) \in X^k$. F has a unique Υ-fixed point.*

3. Main Result

In this section we study applications of multidimensional Υ-fixed point theorem. More precisely applying Theorem 2.3 to a nonlinear integral equation we show the existence and uniqueness of solution. Let $T > 0$ be a real number. Consider the following integral equation on $C([0,T])$:

$$x(t) = \int_0^T \mathcal{G}(t,s)\Big[\sum_{i=1}^{2m} f_i(s,x(s))\Big]ds + p(t), \quad t \in [0,T]. \tag{3.1}$$

Hypothesis for the equation (3.1). We assume:

(a) $f_i : [0,T] \times \mathbb{R} \to \mathbb{R}, \quad 1 \le i \le 2m$ are continuous;

(b) $p : [0,T] \to \mathbb{R}$ is continuous;

(c) $\mathcal{G} : [0,T] \times [0,T] \to [0,\infty)$ is continuous;

(d) there exist positive constants $\eta_1, \eta_2, ..., \eta_{2m}$ such that

$$\max_{1\le i\le 2m} \eta_i \le \Big(2m \max_{0\le t\le T} \int_0^T \mathcal{G}(t,s)ds\Big)^{-1}$$

for all $1 \le i \le 2m$ and

$$0 \le f_{2i-1}(s,y) - f_{2i-1}(s,x) \le \eta_{2i-1} \log^2\Big(1 + \sqrt{y-x}\Big),$$

$$-\eta_{2i} \log^2\Big(1 + \sqrt{y-x}\Big) \le f_{2i}(s,y) - f_{2i}(s,x) \le 0$$

for all $x, y \in \mathbb{R}, \ \ y \ge x$ and $1 \le i \le m$.

(e) there exist continuous functions $y_1^0, y_2^0, \dots, y_{2m}^0 : [0,T] \to \mathbb{R}$ such that $y_{2r-1}^0(t) \le H_{2r-1}(t)$, $1 \le r \le m$ and $y_{2r}^0(t) \ge H_{2r}(t)$, $1 \le r \le m$ for all $t \in [0,T]$ where

$$H_1(t) = \int_0^T \mathcal{G}(t,s)\Big[\sum_{i=1}^{2m} f_i(s,y_i^0(s))\Big]ds + p(t)$$

and

$$H_r(t) = \int_0^T \mathcal{G}(t,s)\Big[\sum_{i=1}^{2m-r+1} f_i(s,y_{i+r-1}^0(s)) + \sum_{\ell=0}^{r-2} f_{2m-\ell}(s,y_{r-1-\ell}^0(s))\Big]ds + p(t),$$

for $2 \le r \le 2m$.

We are ready to formulate our first result.

Theorem 3.1. *Under assumptions (a)-(e), equation (3.1) has a unique solution in* $C[0,T]$.

Proof. As we have mentioned above the proof of this theorem based on to application of Theorem 2.3. To apply this theorem, first we define some necessary notions. We consider the space $X = C[0,T]$ of continuous real functions defined on $[0,T]$ endowed with the standard metric given by

$$d(u,v) = \max_{0\le t\le T} \mid u(t) - v(t) \mid, \quad \text{for} \quad u, v \in X.$$

We define a partial order $\preceq$ on as follows: for any $x, y \in C[0,T]$ we say

$$x \preceq y \Leftrightarrow x(t) \le y(t), \quad \text{for all} \quad t \in [0,T].$$

Let $\Lambda_{2m} = \{1, 2, \dots, 2m\}$. Consider a partition

$$\mathcal{A} = \{1, 3, 5, \dots, 2m-1\} \text{ and } \mathcal{B} = \{2, 4, 6, \dots, 2m\}.$$

We choose $\Upsilon = (\sigma_1, \sigma_2, \dots, \sigma_{2m})$ as follows:

$$\Upsilon = \begin{pmatrix} \sigma_1(1) & \sigma_1(2) & \dots & \sigma_1(2m) \\ \sigma_2(1) & \sigma_2(2) & \dots & \sigma_2(2m) \\ \sigma_3(1) & \sigma_3(2) & \dots & \sigma_3(2m) \\ \vdots & \vdots & \ddots & \vdots \\ \sigma_{2m}(1) & \sigma_{2m}(2) & \dots & \sigma_{2m}(2m) \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 2 & \dots & 2m-2 & 2m-1 & 2m \\ 2 & 3 & \dots & 2m-1 & 2m & 1 \\ 3 & 4 & \dots & 2m & 1 & 2 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 2m & 1 & \dots & 2m-3 & 2m-2 & 2m-1 \end{pmatrix}$$

Next we consider the operator $\mathbb{A} : X^{2m} \to X$

$$\mathbb{A}(\mathbf{x}) = \mathbb{A}(x_1, x_2, \dots, x_{2m}) = \int_0^T \mathcal{G}(t,s)\Big[\sum_{i=1}^{2m} f_i(s, x_i(s))\Big] ds + p(t),$$

where $t \in [0,T]$ and $\mathbf{x} = (x_1, x_2, \dots, x_{2m}) \in X^{2m}$. Further, we show $\mathbb{A}$ satisfies all conditions of Theorem 2.3. Let $\mathbf{x} = (x_1, x_2, \dots, x_{2m})$, $\mathbf{z} = (z_1, z_2, \dots, z_{2m}) \in X^{2m}$. We define a metric in X^{2m} as follows:

$$\mathbf{d}_{2m}(\mathbf{x}, \mathbf{z}) = \max_{i\in\Lambda_{2m}} \{d(x_i, z_i)\} = \max_{i\in\Lambda_{2m}} \{\max_{0\le t\le T} \mid x_i(t) - z_i(t) \mid\}.$$

It is easy to see that $\mathbb{A} : X^{2m} \to X$ is continuous. We show that $\mathbb{A}$ has mixed monotone property w.r.t $\{\mathcal{A}, \mathcal{B}\}$. Let $i \in \mathcal{A}$ i.e i is odd number. Then any $x, y \in X$ with $x \preceq y$ we have

$$\mathbb{A}(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_{2m}) - \mathbb{A}(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_{2m}) = \int_0^T \mathcal{G}(t,s)(f_i(s, y(s)) - f_i(s, x(s)))ds. \tag{3.2}$$

Due to assumptions (**c**) and (**d**) we have $f_i(s, y(s)) - f_i(s, x(s)) \geq 0$ and $\mathcal{G}(t,s)[f_i(s, y(s)) - f_i(s, x(s))] \geq 0$. Thus

$$\mathbb{A}(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_{2m}) \geq \mathbb{A}(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_{2m}).$$

Similarly it can be shown that for $i \in \mathcal{B}$ and for any $x, y \in X$ with $x \preceq y$ we have

$$\mathbb{A}(x_1, \ldots, x_{i-1}, y, x_{i+1}, \ldots, x_{2m}) \leq \mathbb{A}(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_{2m}).$$

Next, we show

$$y_i^0 \preceq \mathbb{A}(y^0_{\sigma_i(1)}, y^0_{\sigma_i(2)}, \ldots, y^0_{\sigma_i(2m)}) \text{ if } i \in \mathcal{A}$$

and

$$y_i^0 \succeq \mathbb{A}(y^0_{\sigma_i(1)}, y^0_{\sigma_i(2)}, \ldots, y^0_{\sigma_i(2m)}) \text{ if } i \in \mathcal{B}.$$

Let $i \in \mathcal{A}$ and $i \geq 3$ (the case $i = 1$ is clear). By assumption (**e**) we have

$$\begin{aligned} y_i^0 \leq H_i &= \int_0^T \mathcal{G}(t,s)\Big[\sum_{k=1}^{2m-i+1} f_k(s, y^0_{k+i-1}(s)) + \sum_{\ell=0}^{i-2} f_{2m-\ell}(s, y^0_{i-1-\ell}(s))\Big]ds + p(t) \\ &= \int_0^T \mathcal{G}(t,s)\Big[f_1(s, y_i^0(s)) + f_2(s, y^0_{i+1}(s)) + \ldots + f_{2m-i+1}(s, y^0_{2m}(s)) \\ &\qquad + f_{2m-i+2}(s, y_1^0(s)) + \ldots + f_{2m}(s, y^0_{i-1}(s))\Big]ds + p(t) \\ &= \mathbb{A}(y_i^0, y^0_{i+1}, \ldots, y^0_{2m}, y^0_1, y^0_2, \ldots, y^0_{i-1}) = \mathbb{A}(y^0_{\sigma_i(1)}, y^0_{\sigma_i(2)}, \ldots, y^0_{\sigma_i(2m)}). \end{aligned}$$

The case $i \in \mathcal{B}$ can be shown similarly. Further, we show that $\mathbb{A}$ satisfies the first condition of Theorem 2.3 with

$$\psi(x) = \sqrt{x}, \quad \theta(x) = \log(1 + \sqrt{x}) \text{ and } \varphi(x) = 0.$$

Let $\mathbf{x} = (x_1, x_2, \ldots, x_{2m}), \mathbf{z} = (z_1, z_2, \ldots, z_{2m}) \in X^{2m}$ with $\mathbf{x} \preceq_{2m} \mathbf{z}$ then

$$d(\mathbb{A}(\mathbf{x}), \mathbb{A}(\mathbf{z})) = \max_{1\leq t\leq T} |\mathbb{A}(x_1, x_2, \ldots, x_{2m})(t) - \mathbb{A}(z_1, z_2, \ldots, z_{2m})(t)|$$
$$= \max_{1\leq t\leq T} \Big(\mathbb{A}(z_1, z_2, \ldots, z_{2m})(t) - \mathbb{A}(x_1, x_2, \ldots, x_{2m})(t)\Big).$$

On the other hand from assumption (**d**) it follows that

$$\mathbb{A}(z_1, z_2, \ldots, z_{2m})(t) - \mathbb{A}(x_1, x_2, \ldots, x_{2m})(t) =$$
$$\int_0^T \mathcal{G}(t,s)\Big[\sum_{i=1}^{2m} f_i(s, z_i(s)) - f_i(s, x_i(s))\Big] ds \leq$$
$$2m(\max_{1\leq i\leq 2m} \eta_i)\Big(\max_{0\leq t\leq T} \int_0^T \mathcal{G}(t,s)ds\Big) \cdot \log^2\Big(1 + \sqrt{\mathbf{d}_{2m}(\mathbf{x}, \mathbf{z})}\Big).$$

Hence

$$\sqrt{d(\mathbb{A}(\mathbf{x}), \mathbb{A}(\mathbf{z}))} \leq \log\Big(1 + \sqrt{\mathbf{d}_{2m}(\mathbf{x}, \mathbf{z})}\Big)$$

that is

$$\psi(d(\mathbb{A}(\mathbf{x}), \mathbb{A}(\mathbf{z}))) \leq \theta(\mathbf{d}_{2m}(\mathbf{x}, \mathbf{z})) - \varphi(\mathbf{d}_{2m}(\mathbf{x}, \mathbf{z})).$$

It is clear $\psi(x) - \theta(x) + \varphi(x) = \sqrt{x} - \log(1 + \sqrt{x}) > 0$ for all $x > 0$. Thus we have shown that the operator $\mathbb{A}$ satisfies the conditions $(i) - (iv)$ of Theorem 2.3. Hence $\mathbb{A}$ has a Υ-fixed point $\mathbf{x}^* = (x_1^*, x_2^*, \ldots, x_{2m}^*)$. That is

$$\begin{aligned} \mathbb{A}(x_1^*, x_2^*, x_3^*, \ldots, x_{2m}^*) &= x_1^*, \\ \mathbb{A}(x_2^*, x_3^*, \ldots, x_{2m}^*, x_1^*) &= x_2^*, \\ &\vdots \\ \mathbb{A}(x_{2m}^*, x_1^*, \ldots, x_{2m-1}^*) &= x_{2m}^*. \end{aligned}$$

It is obvious, for any $\mathbf{x} = (x_1, x_2, \ldots, x_{2m})$, $\mathbf{y} = (y_1, y_2, \ldots, y_{2m}) \in X^{2m}$ there exists a $\mathbf{q} = (q_1, q_2, \ldots, q_{2m}) \in X^{2m}$ such that $\mathbf{x} \preceq_{2m} \mathbf{q}$ and $\mathbf{y} \preceq_{2m} \mathbf{q}$. Indeed, consider the functions $q_i : [0, T] \to \mathbb{R}$

$$q_i(s) = \max\{x_i(s), y_i(s)\},\ s \in [0, T].$$

Since $x_i(s)$ and $y_i(s)$ are continuous on $[0, T]$, the functions $q_i(s)$ are continuous on $[0, T]$ and $x_i(s) \leq q_i(s), y_i(s) \leq q_i(s)$ for all $1 \leq i \leq 2m$. Therefore $\mathbb{A}$ has a unique Υ-fixed point $x^* = (x_1^*, x_2^*, \ldots, x_{2m}^*)$. Next we show

$$x_1^* = x_2^* = \ldots = x_{2m}^*.$$

If $x^* = (x_1^*, x_2^*, \ldots, x_{2m}^*)$ is the Υ-fixed point of $\mathbb{A}$, then $y^* = (y_1^*, y_2^*, \ldots, y_{2m}^*)$ is also a Υ-fixed point of $\mathbb{A}$, where $y_i^* = x_{i+1}^*$ $1 \leq i \leq 2m-1$ and $y_{2m}^* = x_1^*$. However, $\mathbb{A}$ has the unique Υ-fixed point. Therefore $\mathbf{x}^* = \mathbf{y}^*$ hence

$$x_1^* = x_2^* = \ldots = x_{2m}^*.$$

Finally, we have shown that there exists a continuous function $x^*(t)$ such that

$$x^*(t) = \mathbb{A}(x^*, x^*, \ldots, x^*)(t) = \int_0^T \mathcal{G}(t,s)\Big[\sum_{i=1}^{2m} f_i(s, x^*(s))ds\Big] + p(t).$$

This proves Theorem 3.1. □

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