Multidimensional Fixed-Point Theorems and Applications

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Abstract: The purpose of this work is to present the applications of multidimensional fixed point theorems. For this, we prove some multidimensional fixed point theorems and then using these theorems we show the existence and uniqueness of solution of a systems of matrix equations.

1 INTRODUCTION

The concept of two dimensional fixed point (*coupled fixed point*) was introduced by Guo and Lakshmikantham [9] in 1987 and obtained some related results for certain type of mappings. Some twenty years later Bhaskar and Lakshmikantham [8] reconsidered this concept in the context of partially ordered sets by defining the notion of *mixed monotone property* of the mappings. Because of wide applications of these notions many researchers have paid attention and obtained considerable results regarding these concepts. (see, for instance [1], [5], [6], [14], [18]). The concepts of *triple* and *quadruple fixed point* were introduced almost at the same time by Berine and Borcut [3, 4] and Karapinar *et. al.* [10]-[12] and proved some related theorems. The last remarkable result of this direction was obtained by Rolda'n *et. al.* [16, 17]. They have introduced the notion of multidimensional Υ -*fixed point*. This notion covers the concepts of coupled, tripled, quadruple fixed point etc. Due to wide potential application of fixed point results in various branches of mathematics, such as differential equations, mathematical economics, game theory, dynamics, optimal control, functional analysis, operator theory etc. in this work we focus to applications of multidimensional fixed points. We present two applications of multidimensional fixed point theorems. In our first theorem we prove the existence and uniqueness of solution of a nonlinear integral equation by using multidimensional fixed point methods. In our second theorem again applying multidimensional fixed point methods we prove the existence and uniqueness of solution of nonlinear systems of matrices. Finally, in we provide an explicit form of error for the solution of a nonlinear system of matrices.

2 PRELIMINARIES

In this section we recall some necessary notions in order to formulate our main results. These notions can also be found in [16, 17]. Here and further we denote by $(X,d,\prec)$ a *partially ordered metric space*.

Definition 2.1. An ordered metric space $(X,d,\prec)$ is called *regular* if it satisfies the following:

- if $\{x_m\}$ is a nondecreasing sequence and $\{x_m\}\xrightarrow{d} x$ then $x_m \prec x$ for all m ;
- if $\{y_m\}$ is a nondecreasing sequence and $\{y_m\}\xrightarrow{d} y$ then $y_m \succ y$ for all m .

Taking a natural number $k\geq 2$ we consider the set $\Lambda_k=\{1,2,...,k\}$. Let $\{A,B\}$ be a partition of Λ_k that is $A\cup B=\Lambda_k$ and $A\cap B=\varnothing$. Using this partition and partially ordered metric space $(X,d,\prec)$ we define a k -dimensional partially ordered metric space $(X^k,d_k,\prec_k)$ as follows:

- the k -cartesian power of a set X

$$X^k = \underbrace{X \times X \times ... \times X}_{k-times}$$

- the maximum metric $\mathbf{d}_k : X^k \times X^k \to [0,+\infty)$ given by
$$\mathbf{d}_k(\mathbf{x},\mathbf{y}) = \max_{1\le i\le k}\{d_k(x_i, y_i)\}$$
where $\mathbf{x} = (x_1, x_2, ..., x_k)$ and $\mathbf{y} = (y_1, y_2, ..., y_k)$.

- the partial order w.r.t $\{A, B\}$ that is, for any $\mathbf{x} = (x_1, x_2, ..., x_k)$ and $\mathbf{y} = (y_1, y_2, ..., y_k)$ we have
$$\mathbf{x} \prec_k \mathbf{y} \Leftrightarrow \begin{cases} x_i \prec y_i \text{ if } i \in A, \\ x_i \succ y_i \text{ if } i \in B. \end{cases}$$

Definition 2.2. We say that a mapping $F : X^k \to X^k$ has the *mixed monotone* property w.r.t partition $\{A, B\}$ if F is monotone nondecreasing in arguments of A and monotone nonincreasing in arguments of B.

We define the following set of mappings:

$$\Omega_{A,B} = \{\sigma : \Lambda_k \to \Lambda_k : \sigma(A) \subseteq A, \sigma(B) \subseteq B\},$$

$$\Omega'_{A,B} = \{\sigma : \Lambda_k \to \Lambda_k : \sigma(A) \subseteq B, \sigma(B) \subseteq A\}.$$

Let $\Upsilon = (\sigma_1, \sigma_2, ..., \sigma_k)$ be k-tuple of mappings of $\sigma_i : \Lambda_k \to \Lambda_k$ such that $\sigma_i \in \Omega_{A,B}$ if $i \in A$ and $\sigma_i \in \Omega'_{A,B}$ if $i \in B$. In the sequel we consider only such kind of k-tuple of mappings.

Definition 2.3. A point $\mathbf{x} = (x_1, x_2, ..., x_k) \in X^k$ is called Υ*-fixed point* of a mapping $F : X^k \to X$ if

$$F(x_{\sigma_i(1)}, x_{\sigma_i(2)}, ..., x_{\sigma_i(k)}) = x_i$$

for all $i \in \Lambda_k$.

3 NECESSARY MULTIDIMENSIONAL FIXED POINT THEOREMS

In this section we provide relations between one and multidimensional fixed point theorems.

Define $T_\Upsilon : X^k \to X^k$ as follows:

$$T_\Upsilon(x_1, x_2, ..., x_k) = (F(x_{\sigma_1(1)}, x_{\sigma_1(2)}, ..., x_{\sigma_1(k)}), F(x_{\sigma_2(1)}, x_{\sigma_2(2)}, ..., x_{\sigma_2(k)}), ..., F(x_{\sigma_k(1)}, x_{\sigma_k(2)}, ..., x_{\sigma_k(k)}))$$

for all $\mathbf{x} = (x_1, x_2, ..., x_k) \in X^k$. The following theorem was obtained Roldan's.

Theorem 3.1. [17] Let $(X, d, \prec)$ be a complete partially ordered metric space. Let $\Upsilon : \Lambda_k \to \Lambda_k$ be a k-tuple of mappings $\Upsilon = (\sigma_1, \sigma_2, ..., \sigma_k)$ which is verifying $\sigma_i \in \Omega_{A,B}$ if $i \in A$ and $\sigma_i \in \Omega'_{A,B}$ if $i \in B$.

- If F has the mixed monotone property, then T_Υ is monotone nondecreasing w.r.t $\prec_k$.
- If F is continuous, then T_Υ is also continuous.

- A point $\mathbf{x}=(x_1,x_2,...,x_k)\in X^k$ is a Υ-fixed point of F, if and only if $\mathbf{x}=(x_1,x_2,...,x_k)$ is a fixed point of T_Υ.

We need the following definition which was introduced by Khan *et. al.* in [13].

Definition 3.2. A function $\psi:[0,+\infty)\to[0,+\infty)$ is called *altering distance function*, if ψ is continuous, monotonically increasing and $\psi(0)=0$.

Theorem 3.3. [17] Let $(X,d,\prec)$ be a complete partially ordered metric space. Let $\Upsilon:\Lambda_k\to\Lambda_k$ be a k-tuple of mappings $\Upsilon=(\sigma_1,\sigma_2,...,\sigma_k)$ which is verifying $\sigma_i\in\Omega_{A,B}$ if $i\in A$ and $\sigma_i\in\Omega'_{A,B}$ if $i\in B$. Suppose a continuous $F:X^k\to X$ satisfies the following conditions:

i. there exist altering distance functions ψ,φ such that for all $\mathbf{x}=(x_1,x_2,...,x_k)$, $\mathbf{y}=(y_1,y_2,...,y_k)$ with $\mathbf{x}\prec_k\mathbf{y}$

$$\psi(d(F(x_1,x_2,...,x_k),F(y_1,y_2,...,y_k)))\le\psi(\mathbf{d}_k(\mathbf{x},\mathbf{y}))-\varphi(\mathbf{d}_k(\mathbf{x},\mathbf{y}));$$

ii. there exists $\mathbf{x}^0=(x_1^0,x_2^0,...,x_k^0)$ such that $x_i^0\prec F(x^0_{\sigma_i(1)},x^0_{\sigma_i(2)},...,x^0_{\sigma_i(k)})$ if $i\in A$ and

$x_i^0\succ F(x^0_{\sigma_i(1)},x^0_{\sigma_i(2)},...,x^0_{\sigma_i(k)})$ if $i\in B$;

iii. F has the mixed monotone property w.r.t $\{A,B\}$;

iv. For all $i\in\Lambda_k$ the mapping σ_i is a permutation of Λ_k;

Then F has at least one Υ-fixed point. Moreover, if for any $\mathbf{x}=(x_1,x_2,...,x_k)$, $\mathbf{y}=(y_1,y_2,...,y_k)$ there exists a $\mathbf{z}=(z_1,z_2,...,z_k)$, such that $\mathbf{x}\prec_k\mathbf{z}$ and $\mathbf{y}\prec_k\mathbf{z}$, then F has a unique Υ-fixed point.

Note that, in above theorem the authors required to be a permutation of the mapping σ_i for all $i\in\Lambda_k$ (i.e. condition iv.) It runs out this condition would not be necessary, if we change the contractive condition of Theorem 3.3 as follows:

Theorem 3.4 Let $(X,d,\prec)$ be a complete partially ordered metric space. Let $\Upsilon:\Lambda_k\to\Lambda_k$ be a k-tuple of mappings $\Upsilon=(\sigma_1,\sigma_2,...,\sigma_k)$ which is verifying $\sigma_i\in\Omega_{A,B}$ if $i\in A$ and $\sigma_i\in\Omega'_{A,B}$ if $i\in B$. Suppose a continuous $F:X^k\to X$ satisfies the conditions ii.-iv. and

a) there exist altering distance functions ψ,θ and a monotonically decreasing continuous function $\varphi:[0,\infty)\to R$ such that for all $\mathbf{x}=(x_1,x_2,...,x_k)$, $\mathbf{y}=(y_1,y_2,...,y_k)$ with $\mathbf{x}\prec_k\mathbf{y}$

$$\psi(d(F(x_1,x_2,...,x_k),F(y_1,y_2,...,y_k)))\le\theta(\mathbf{d}_k(\mathbf{x},\mathbf{y}))-\varphi(\mathbf{d}_k(\mathbf{x},\mathbf{y}))$$

where $\theta(0)=\varphi(0)=0$ and $\psi(x)-\theta(x)+\varphi(x)>0$ for all $x>0$.

Then F has at least one Υ-fixed point. Moreover, if for any $\mathbf{x}=(x_1,x_2,...,x_k)$, $\mathbf{y}=(y_1,y_2,...,y_k)$ there exists a $\mathbf{z}=(z_1,z_2,...,z_k)$, such that $\mathbf{x}\prec_k\mathbf{z}$ and $\mathbf{y}\prec_k\mathbf{z}$, then F has a unique Υ-fixed point.

Proof. Using condition i*. we get

$$\psi(\mathbf{d}_k(T_\Upsilon(\mathbf{x}),T_\Upsilon(\mathbf{y}))=\psi(\max_{i\in\Lambda_k} d(F(x_{\sigma_i(1)},x_{\sigma_i(2)},...,x_{\sigma_i(k)}),F(y_{\sigma_i(1)},y_{\sigma_i(2)},...,y_{\sigma_i(k)}))$$
$$\le \max_{i\in\Lambda_k}(\theta(\max_{j\in\Lambda_k} d(x_{\sigma_i(j)},y_{\sigma_i(j)}))-\varphi(\max_{j\in\Lambda_k} d(x_{\sigma_i(j)},y_{\sigma_i(j)}))$$
$$\le \theta(\mathbf{d}_k(\mathbf{x},\mathbf{y}))-\varphi(\mathbf{d}_k(\mathbf{x},\mathbf{y}))$$

for all $\mathbf{x}=(x_1,x_2,...,x_k)$, $\mathbf{y}=(y_1,y_2,...,y_k)$ such that $\mathbf{x}\prec_k \mathbf{y}$. Thus we have shown that the mapping T_Υ satisfies the contractive condition of Theorem 2.5 in [18]. The rest of the proof follows exactly same way that of Theorem 2.5 in [18]. □

4 MAIN RESULTS

In this section we study applications of multidimensional Υ-fixed point theorem. We provide two applications of multidimensional Υ-fixed point theorem and at the end we analysis the error for the provided application. More precisely, applying Theorem 3.5 to systems of matrix equations we show the existence and uniqueness of solution and estimate the error of approximation of high iterations of matrices.

We deal on the set of $n\times n$ matrices and we denote this set by $M(n)$. Let $H(n)$ be the set of all $n\times n$ Hermitian matrices, $P(n)$ be the set of all $n\times n$ positive definite matrices and $\tilde{P}$ be the set of all $n\times n$ positive semidefinite matrices. We study the existence and uniqueness of solution (X_1,X_2,X_3) to the system of matrix equations

$$\begin{cases} X_1=Q+A_1^*\Phi(X_1)A_1-A_2^*\Phi(X_2)A_2+A_3^*\Phi(X_3)A_3\\ X_2=Q+A_1^*\Phi(X_2)A_1-A_2^*\Phi(X_1)A_2+A_3^*\Phi(X_2)A_3\\ X_3=Q+A_1^*\Phi(X_3)A_1-A_2^*\Phi(X_2)A_2+A_3^*\Phi(X_1)A_3 \end{cases} \tag{1}$$

where $Q\in P(n),\ X_1,X_2,X_3\in H(n),\ A_1,A_2,A_3\in M(n)$ and $\Phi: H(n)\to H(n).$

Let us first define some necessary facts. A partial order $\prec$ on $H(n)$ is defined by

$$X,Y\in H(n),\ \ X\prec Y \Leftrightarrow Y-X\in\tilde{P}.$$

The set $H(n)$ is partially ordered and for every $X,Y\in H(n)$ there is a greatest lower bound and a least upper bound (see [2]). Next we use the following two norms:

- $\|A\|=\sqrt{\lambda_{\max}(A^*A)}$ the spectral norm;
- $\|A\|_1=tr(\sqrt{A^*A})$ the trace norm.

Further it is convenient us to use metric induced by the trace norm. Since $H(n)$ is a finite dimensional linear metric space equipped the metric indicate by $\|\cdot\|_1$ is complete (see Theorem IX.2.2 in [7]). The following lemma plays a key role for our application.

Lemma 4.1 (see [2]) Let $A,B\in\tilde{P}(n)$. Then we have

$$0\le tr(AB)\le \|A\|\,tr(B)$$

where $\|\cdot\|$ is the spectral norm.

Here and below we suppose:

a) $A_2^*\Phi(Q)A_2 - Q \in P(n)$;

b) Φ is continuous, $\Phi(0_n) = 0_n$ and preserves the order, where 0_n is the $n \times n$ zero matrix;

c) there exists a positive number M such that

$$A_1A_1^* + A_2A_2^* + A_3A_3^* - MI_n \in P(n)$$

where $I_n \in M(n)$ is the identity matrix;

d) for any $X, Y \in H(n)$ such that $Y \prec X$ we have

$$\left| tr\left(\Phi(X) - \Phi(Y)\right)\right| \le \frac{1}{M}\exp\left(-\frac{1}{tr(X-Y)}\right)$$

where $tr(\cdot)$ is the trace of matrix.

We are ready to formulate our first result.

Theorem 4.2. Under assumptions a)-d), the system of equations (1) has a unique solution in $H(n)$.

Proof. Let $\Lambda_3 = \{1,2,3\}$. Consider a partition $A = \{1,3\}$ and $B = \{2\}$. We choose $\Upsilon = (\sigma_1, \sigma_2, \sigma_3)$ as

$$\Upsilon = \begin{pmatrix} \sigma_1(1) & \sigma_1(2) & \sigma_1(3) \\ \sigma_2(1) & \sigma_2(2) & \sigma_2(3) \\ \sigma_3(1) & \sigma_3(2) & \sigma_3(3) \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 2 \\ 3 & 2 & 1 \end{pmatrix}$$

Next we consider the operator $\mathbf{D}: H^3(n) \to H(n)$

$$\mathbf{D}(X_1, X_2, X_3) = Q + A_1^*\Phi(X_1)A_1 - A_2^*\Phi(X_2)A_2 + A_3^*\Phi(X_3)A_3. \qquad (2)$$

It is clear the system (1) has a solution if and only if $\mathbf{D}$ has a Υ - fixed point. Therefore, further we show that the operator $\mathbf{D}$ satisfies all conditions of Theorem 3.4. Since Φ is continuous, $\mathbf{D}$ is continuous. Next we show that $\mathbf{D}$ has the mixed monotone property w.r.t $\{A, B\}$. By assumption b) the mapping Φ preserves order, therefore for any $(X_1, X_2, X_3), (Y_1, Y_2, Y_3) \in H(n)$ such that

$$(X_1, X_2, X_3) \prec_3 (Y_1, Y_2, Y_3) \Leftrightarrow \begin{cases} X_1 \prec Y_1 \\ X_2 \succ Y_2 \\ X_3 \prec Y_3 \end{cases}$$

we have

$$\left(\Phi(X_1), \Phi(X_2), \Phi(X_3)\right) \prec_3 \left(\Phi(Y_1), \Phi(Y_2), \Phi(Y_3)\right) \Leftrightarrow \begin{cases} \Phi(X_1) \prec \Phi(Y_1) \\ \Phi(X_2) \succ \Phi(Y_2) \\ \Phi(X_3) \prec \Phi(Y_3). \end{cases}$$

Thus

$$\mathbf{D}(Y_1, Y_2, Y_3) - \mathbf{D}(X_1, X_2, X_3) = A_1^*\left(\Phi(Y_1) - \Phi(X_1)\right)A_1 + A_2^*\left(\Phi(X_1) - \Phi(Y_2)\right)A_2 + A_3^*\left(\Phi(Y_3) - \Phi(X_3)\right)A_3 \succ 0_n.$$

Let $(Z_1^0, Z_2^0, Z_3^0) = (Q, 0_n, Q)$ next we show

$$Z_i^0 \prec_i \mathbf{D}(Z_{\sigma_i(1)}^0, Z_{\sigma_i(2)}^0, Z_{\sigma_i(3)}^0) \quad \text{for all } i \in \Lambda_3.$$

Indeed

$$Q \prec \mathbf{D}(Q, 0_n, Q) = Q + A_1^* \Phi(Q) A_1 + A_3^* \Phi(Q) A_3$$

and by assumption a) we have

$$\mathbf{D}(0_n, Q, 0_n) = Q - A_2^* \Phi A_2 \succ 0_n .$$

Further, we show that $\mathbf{D}$ satisfies the first condition of Theorem 3.4 with

$$\psi(x) = e^{-1/x} \ \ x > 0, \ \psi(0) = 0, \ \theta(x) = C\psi(x)$$

for some $C \in (0,1)$ and $\varphi(x) = 0$. Let $(X_1, X_2, X_3), (Y_1, Y_2, Y_3) \in H(n)$ such that $(X_1, X_2, X_3) \prec_3 (Y_1, Y_2, Y_3)$. Because of $\mathbf{D}(X_1, X_2, X_3) \prec \mathbf{D}(Y_1, Y_2, Y_3)$ we have

$$\begin{aligned} \| \mathbf{D}(X_1, X_2, X_3) - \mathbf{D}(Y_1, Y_2, Y_3) \|_1 &= tr\left(A_1^* \left(\Phi(Y_1) - \Phi(X_1)\right) A_1\right) \\ & tr\left(A_2^* \left(\Phi(X_1) - \Phi(Y_2)\right) A_2\right) + tr\left(A_3^* \left(\Phi(Y_3) - \Phi(X_3)\right) A_3\right) \\ &\leq \sum_{i=1}^{3} \| A_i A_i^* \| \| \Phi(X_i) - \Phi(Y_i) \|_1 . \end{aligned} \tag{3}$$

Applying assumption d) we get

$$\sum_{i=1}^{3} \| A_i A_i^* \| \| \Phi(X_i) - \Phi(Y_i) \|_1 \leq \frac{\sum_{i=1}^{3} \| A_i A_i^* \|}{M} \exp\left(- \frac{1}{\max_{1 \leq i \leq 3} \| X_i - Y_i \|_1} \right). \tag{4}$$

It is obvious $\exp(-1/x) < x$ for $x > 0.$ Taking into account this and inequality (3) and (4) we get

$$\exp\left(- \frac{1}{\| \mathbf{D}(X_1, X_2, X_3) - \mathbf{D}(Y_1, Y_2, Y_3) \|_1} \right) \leq \frac{\sum_{i=1}^{3} \| A_i A_i^* \|}{M} \exp\left(- \frac{1}{\max_{1 \leq i \leq 3} \| X_i - Y_i \|_1} \right).$$

Assumption c) implies

$$\frac{\sum_{i=1}^{3} \| A_i A_i^* \|}{M} < 1.$$

On the other hand, for all $X, Y \in H(n)$ there is a greatest lower bound and least upper bound. Therefore we have shown the operator $\mathbf{D}$ satisfies all the conditions of Theorem 3.4. Hence $\mathbf{D}$ has a unique Υ - fixed point $(X_1^*, X_2^*, X_3^*) \in H^3(n)$ which is also the unique solutions of the system (1), that is

$$\begin{cases} X_1^* = Q + A_1^*\Phi(X_1^*)A_1 - A_2^*\Phi(X_2^*)A_2 + A_3^*\Phi(X_3^*)A_3 \\ X_2^* = Q + A_1^*\Phi(X_2^*)A_1 - A_2^*\Phi(X_1^*)A_2 + A_3^*\Phi(X_2^*)A_3 \\ X_3^* = Q + A_1^*\Phi(X_3^*)A_1 - A_2^*\Phi(X_2^*)A_2 + A_3^*\Phi(X_1^*)A_3 \end{cases}$$

$\square$

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REFERENCES

1. M. Abbas, A.R. Khan, T. Nazir, "Coupled common fixed point results in two generalized metric spaces", *Appl. Math. Comput.*, 217, no. 13, 6328-6336, (2011).
2. Andre C. M. Ran and Martine C. B. Reurings. "A fixed point theorem in partially ordered sets and some applications to matrix equations," *Proc. Amer. Math. Soc.* Volume 132, no. 5, pp. 1435-1443, (2003).
3. V. Berinde, M. Borcut , "Tripled fixed point theorems for contractive type mappings in partiallyordered matric space", *Nonlinear Analysis. Theory, Methods \& Applications,* 74, 4889-4897, (2011).
4. M. Borcut, V. Berinde, "Tripled coincidence theorems for contractive type mappings in partiallyordered matric space", *Appl. Math. Comput*, 218, pp. 5929-5936, (2012).
5. B.S. Choudhury and A. Kundu, "(ψ,α,β)-weak contractions in partially ordered metric spaces". *Applied Mathematics Letters,* 25, pp. 6-10, (2012).
6. D. Doric, Z. Kadelburg and S. Radenovic, "Coupled fixed point results for mappings without mixed monotone property", *Applied Mathematics Letters* 25, pp. 1803-1808, (2012).
7. I. Gohberg and S. Goldberg, "Besic Operator Theory," Birkhauser, Boston, MA, 1981. MR 83b, 47001.
8. T. G. Bhaskar and V. Lakshmikantham, "Fixed point theorems in partially ordered metric space and applications,"*Nonlinear Analysis. Theory, Methods \& Applications,* vol. 65, no. 7, pp. 1379-1393, (2006).
9. D. J. Guo and V. Lakshmikantham, "Coupled fixed points of nonlinear operators with applications," *Nonlinear Analysis. Theory, Methods \& Applications*, vol. 11, no. 5, pp. 623-632, (1987).
10. E. Karapinar and V. Berinde, "Quadruple fixed point theorems for nonlinear contractions in partially ordered matric space," *Banach Journal of Mathematical Analysis*, vol. 6, no. 1, pp. 74-89, (2012).
11. E. Karapinar and N. V. Luong, "Quadruple fixed point theorems for nonlinear contractions," *Computers \& Mathematics with Applications*, vol. 64, no. 6, pp. 1839-1848, (2012).
12. E. Karapinar and K. Tas, "Quadruple fixed point theorems for nonlinear contractions on partial metric spaces," *Appl. Gen. Topol.*, vol. 15, no. 1, pp. 11-24, (2014).
13. M. S. Khan, M. Swaleh and S. Sessa, "Fixed point theorems by altering distances between the points," *Bulletin of the Australian Mathematical Society*, vol. 30, no. 1, pp. 1-9, (1984).
14. V. Lakshmikantham and L. Ciric, "Coupled fixed point theorems for nonlinear contractions in partially ordered metric spaces," *Nonlinear Analysis. Theory, Methods \& Applications*, vol. 70, pp. 4341-4349, (2009).
15. H. K. Nashine and W. Shatanawi, "Cuopled fixed point theorems for a pair of commuting mappings in partially ordered complete metric space," *Comput. Math. Appl.* 62, 1984-1993, (2011).
16. A. Roldan, J. Martinez-Moreno, and C. Roldan, "Multidimensional fixed point theorems in partially ordered complete metric spaces," *Journal of Mathematical Analysis and Applications*, vol. 396, no. 2, pp. 536-545, (2012).
17. A. Roldan, J. Martinez-Moreno, C. Roldan and E. Karapinar, "Some remarks on multidimensional fixed point theorems", *Fixed Point Theory*, 15, no. 2, 545-558, (2014).
18. F. Shaddad, M. S. Noorani, S. M. Alsulami, and H. Akhadkulov, "Coupled point results in partially ordered metric spaces without compatibility", *Fixed Point Theory and Applications*, 2014:204, (2014).